INTERNATIONAL JOURNAL FOR NUMERICAL AND ANALYTICAL METHODS IN GEOMECHANICS, VOL. 20, 605–613 (1996)

EXACT STRESS INTEGRATION FOR VON MISES ELASTO-PLASTIC MODEL WITH CONSTANT HARDENING MODULUS

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SUMMARY

Exact integration for the elasto-plastic von Mises Yield Criterion is presented for general stress space. Truncated series solution is obtained under a constant strain rate assumption. On the other hand, a trial and error definite integral solution is obtained under the assumption of constant stress rate. Either could be suitable for the numerical implementation of the model on vector computers and speeding of finite element calculations.

KEY WORDS: elasto-plastic von Mises yield criterion: numerical integration scheme

INTRODUCTION

The von Mises failure criterion was introduced in 1913. The failure criterion is used widely in metal and geomechanics material due to its simplicity. It has been used as the vehicle in investigating the accuracy of numerical integration schemes; see e.g. References 1 and 2. In Reference 1, the closed-form solution of the perfectly plastic constitutive equation of von Mises was presented, and to the knowledge of the author, no attempt has been made to the hardening form of the criterion. In most numerical comparisons, the numerical solutions were compared with the alternative numerical solutions of very fine steps. In this paper, an attempt is made to introduce a series solution of the elastoplastic von Mises yield criterion with constant hardening modulus.

The criterion states that the material will yield when the second invariant of the deviatoric stress tensor reaches a certain yield stress or in soil mechanics term, two times the cohesion:

$$\sqrt{J_2'} = 2c \tag{1}$$

where J_2' is the second invariant of deviatoric stress

$$J_2' = \frac{s_{ij}s_{ji}}{2} \tag{2}$$

and deviatoric stress is defined as

$$s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}/3 \tag{3}$$

CCC 0363–9061/96/080605–09

Received 1 November 1995
Revised 5 December 1995

with δ_{ij} the Kronecker delta function. Indicial notation is used throughout this paper and repeated subscripts implies summation from one to the dimensionality of the problem unless otherwise stated.

In the numerical implementation of the constitutive relationship, stress integration techniques have to be employed. Given an initial stress state and a strain increment, one would like to know the final stress state. In usual numerical implementations, the strain increment is subdivided into smaller increments, assumed tangential relationship is used to integrate the stress. Then the stress point is bought back to the yield surface using the consistency equation. As different point in the numerical domain may have a different initial state and strain increment, the amount of strain subdivision within the domain will not be uniform.

This is major problem in applying the technique to a vector or pipeline computer which requires a series of actions to be applied equally to all elements of the vector in order to achieve maximum efficiency. Alternative strategies include the backward Euler formulation, as it is an iterative procedure, the number of iterations to convergence will not be known in advance, thus precluding any vectorization to the scheme.

In this paper, a series solution is presented so that the integration can be carried out in a step and no 'bring back' to the yield surface is required if the incremental step is kept small.

The first assumption used is the common one on the decomposition of incremental strain rate in elastic and plastic component with elastic strain rate defined as

$$\mathrm{d}\sigma_{ij} = D^{e}_{ijkl}\,\mathrm{d}\varepsilon^{e}_{kl} \tag{4}$$

and the elastic D-matrix is as follows:

$$D^{e}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2G\delta_{ik}\delta_{jl} \tag{5}$$

where λ and G are the Lame's constants for elasticity. On the other hand, the plastic strain rate is defined as

$$\mathrm{d}\varepsilon^{p}_{ij} = \frac{1}{H}\,\frac{\dfrac{\partial Q}{\partial\sigma_{ij}}\,\dfrac{\partial F}{\partial\sigma_{kl}}}{\left|\dfrac{\partial Q}{\partial\sigma}\right|\left|\dfrac{\partial F}{\partial\sigma}\right|}\,\mathrm{d}\sigma_{kl} \tag{6}$$

where F is the yield criterion, Q the plastic potential and H is the hardening modulus. In this paper, only constant value of H is being considered. After some algebraic manipulations, the incremental stress rate and the elastoplastic D-matrix can be defined as

$$\mathrm{d}\sigma_{ij} = D^{e}_{ijkl}\,\mathrm{d}\varepsilon_{kl} - \frac{D^{e}_{ijkl}\dfrac{\partial Q}{\partial\sigma_{kl}}\dfrac{\partial F}{\partial\sigma_{ij}}D^{e}_{ijkl}}{H\left|\dfrac{\partial Q}{\partial\sigma}\right|\left|\dfrac{\partial F}{\partial\sigma}\right| + \dfrac{\partial F}{\partial\sigma_{ij}}D^{e}_{ijkl}\dfrac{\partial Q}{\partial\sigma_{kl}}}\,\mathrm{d}\varepsilon_{kl} \tag{7}$$

The rate of plastic work done can be defined as

$$\mathrm{d}W^{p} = \sigma_{ij}\,\mathrm{d}\varepsilon^{p}_{ij} = \frac{\sigma_{ij}}{H}\,\frac{\dfrac{\partial Q}{\partial\sigma_{ij}}\,\dfrac{\partial F}{\partial\sigma_{kl}}}{\left|\dfrac{\partial Q}{\partial\sigma}\right|\left|\dfrac{\partial F}{\partial\sigma}\right|}\,\mathrm{d}\sigma_{kl} \tag{8}$$

The second assumption made is that Q is an order one homogeneous function of stress and equation (8) can be reduced to

$$\mathrm{d}W^{\mathrm{p}} = \sigma_{ij}\mathrm{d}\varepsilon_{ij}^{\mathrm{p}} = \frac{1}{H}\frac{Q}{|\partial Q/\partial \sigma|}\frac{\mathrm{d}F}{|\partial F/\partial \sigma|} \tag{9}$$

The third assumption is that the material obeys an associative flow rule, i.e. $F \equiv Q$.

$$\mathrm{d}W^{\mathrm{p}} = \sigma_{ij}\mathrm{d}\varepsilon_{ij}^{\mathrm{p}} = \frac{1}{H}\frac{F}{|\partial F/\partial \sigma|^2}\mathrm{d}F \tag{10}$$

Lastly, we assume that F is the Von Mises[3] yield criterion, i.e.

$$F = \sqrt{J'2} \tag{11}$$

with

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{s_{ij}}{2\sqrt{J_2'}} \tag{12}$$

and

$$\left|\frac{\partial F}{\partial \sigma}\right|^2 = \frac{1}{2} \tag{13}$$

also for $H =$ constant and non-zero, we have

$$HW^{\mathrm{p}} + J_2'(0) = J_2' \tag{14}$$

where $J_2'(0)$ is the initial value of J_2'. Next, we are going to simplify the elastoplastic D-matrix in equation (5) with the following expressions:

$$D_{ijkl}^{\mathrm{e}}\frac{\partial F}{\partial \sigma_{kl}} = \frac{Gs_{ij}}{\sqrt{J_2'}} \tag{15}$$

$$\frac{\partial F}{\partial \sigma_{ij}}D_{ijkl}^{\mathrm{e}}\frac{\partial F}{\partial \sigma_{kl}} = G \tag{16}$$

The incremental strain rate is also decomposed into volumetric and deviatoric components with

$$\mathrm{d}e_{ij} = \mathrm{d}\varepsilon_{ij} - \delta_{ij}\mathrm{d}\varepsilon_{kk}/3 \tag{17}$$

with rate of deviatoric work done defined as

$$\mathrm{d}W^{\mathrm{s}} = s_{ij}\mathrm{d}\varepsilon_{ij} = s_{ij}\mathrm{d}e_{ij} \tag{18}$$

So the D-matrix in equation (7) becomes

$$\mathrm{d}\sigma_{ij} = D_{ijkl}^{\mathrm{e}}\mathrm{d}\varepsilon_{kl} - \frac{2G^2 s_{ij}s_{kl}\mathrm{d}e_{kl}}{J_2'(H+2G)} \tag{19}$$

Considering only the deviatoric part

$$\mathrm{d}s_{ij} = 2Gs_{ij}\mathrm{d}e_{ij} - \frac{2G^2 s_{ij}s_{kl}\mathrm{d}e_{kl}}{J_2'(H+2G)} \tag{20}$$

premultiplying with s_{ij},

$$s_{ij}\mathrm{d}s_{ij} = 2Gs_{ij}\mathrm{d}e_{ij} - \frac{2G^2 s_{ij}s_{ij}s_{kl}\mathrm{d}e_{kl}}{J'_2(H + 2G)} \tag{21}$$

Using equation (18), we obtain an expression relating the rate of second deviatoric stress invariant with the rate of deviatoric work done:

$$\mathrm{d}J'_2 = \frac{2GH}{H + 2G}\mathrm{d}W^s \tag{22}$$

Putting $k = G/H$ and $\mathrm{d}e_{ij} = \Delta e_{ij}\mathrm{d}t$, i.e. the incremental strain rate is assumed to be constant (this is a common assumption in most finite element software), substituting equation (18) and (22) into equation (20) and introducing the integrating factor $J_2'^k$, we have

$$\frac{\mathrm{d}(J_2'^k s_{ij})}{\mathrm{d}t} = 2G\Delta e_{ij}J_2'^k \tag{23}$$

Premultiplying with Δe_{ij},

$$\frac{\mathrm{d}(J_2'^k s_{ij}\Delta e_{ij})}{\mathrm{d}t} = 2G\Delta e_{ij}\Delta e_{ij}J_2'^k \tag{24}$$

Noting that

$$\mathrm{d}W^s = s_{ij}\mathrm{d}e_{ij} = s_{ij}\Delta e_{ij}\mathrm{d}t \tag{25}$$

and using equation (22), one obtains

$$\frac{\mathrm{d}\left(J_2'^k \dfrac{\mathrm{d}J'_2}{\mathrm{d}t}\right)}{\mathrm{d}t} = \frac{2GH}{H + 2G}2G\Delta e_{ij}\Delta e_{ij}J_2'^k \tag{26}$$

Defining the positive constant

$$A = \frac{4G^2\Delta e_{ij}\Delta e_{ij}(k + 1)}{(2k + 1)} \tag{27}$$

we obtain a second-order ordinary differential equation

$$\frac{\mathrm{d}^2 J_2'^{k+1}}{\mathrm{d}t^2} = AJ_2'^k \tag{28}$$

On the other hand, if we take each s_{ij} as y in equation (20), we have a standard set of first-order ordinary differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t) \tag{29}$$

with analytical solution

$$y = \mathrm{e}^{-\int p\,\mathrm{d}t}\left(\int q\mathrm{e}^{\int p\,\mathrm{d}t}\mathrm{d}t + \mathrm{const}\right) \tag{30}$$

For equation (20) using equation (22), we have

$$p(t) = \frac{k}{J_2'}\frac{\mathrm{d}J_2'}{\mathrm{d}t} \tag{31}$$

so

$$\int p\,\mathrm{d}t = k\ln J_2' \tag{32}$$

and

$$\mathrm{e}^{-\int p\,\mathrm{d}t} = J_2'^{-k} \tag{33}$$

Furthermore, using equation (15), we have

$$\int q\mathrm{e}^{\int p\,\mathrm{d}t}\,\mathrm{d}t = \int 2G\Delta e_{ij}J_2'^{k}\,\mathrm{d}t = 2G\Delta e_{ij}\frac{\mathrm{d}J_2'^{k+1}}{A\,\mathrm{d}t} \tag{34}$$

so we obtain an expression for the variation of deviatoric stress with C_{ij} being the integration constant for each component of the stress tensor:

$$s_{ij} = \frac{2G\Delta e_{ij}(k+1)}{A}\frac{\mathrm{d}J_2'}{\mathrm{d}t} + \frac{C_{ij}}{J_2'^{k}} \tag{35}$$

or

$$s_{ij} = \frac{\Delta e_{ij}(2k+1)}{2G\Delta e_{ij}\Delta e_{ij}}\frac{\mathrm{d}J_2'}{\mathrm{d}t} + \frac{C_{ij}}{J_2'^{k}} \tag{36}$$

Considering

$$s_{ij}\Delta e_{ij} = \frac{\Delta e_{ij}\Delta e_{ij}(2k+1)}{2G\Delta e_{ij}\Delta e_{ij}}\frac{\mathrm{d}J_2'}{\mathrm{d}t} + \frac{C_{ij}\Delta e_{ij}}{J_2'^{k}} \tag{37}$$

we have, using (22),

$$\frac{2k+1}{2G}\frac{\mathrm{d}J_2'}{\mathrm{d}t} = \frac{2k+1}{2G}\frac{\mathrm{d}J_2'}{\mathrm{d}t} + \frac{C_{ij}\Delta e_{ij}}{J_2'^{k}} \tag{38}$$

and this implies

$$c_{ij}\Delta e_{ij} = 0 \tag{39}$$

There should be two independent solutions for equation (28) and one can be found by substituting

$$J_2' = (at+b)^2 \tag{40}$$

this gives

$$a^2 = \frac{A}{(2k+2)(2k+1)} \tag{41}$$

and this solution will give $C_{ij}C_{ij} = 0$ (see below) when the identity $2J'_2 = s_{ij}s_{ij}$ is considered for (36) and (40) implying that the stress state is always proportional to the incremental strain. This can only be considered as a special solution for the equation with intial deviatoric stresses proportional to the incremental strain

$$\begin{aligned}2J'_2 = s_{ij}s_{ij} &= \frac{2G\Delta e_{ij}(k+1)}{A}\frac{2G\Delta e_{ij}(k+1)}{A}4a^2(at+b)^2 + \frac{C_{ij}C_{ij}}{(at+b)^{4k}}\\ &= \frac{2G\Delta e_{ij}(k+1)}{A}\frac{2G\Delta e_{ij}(k+1)}{2(k+1)(2k+1)}4(at+b)^2 + \frac{C_{ij}C_{ij}}{(at+b)^{4k}}\\ &= \frac{2G\Delta e_{ij}(k+1)}{A}\frac{G\Delta e_{ij}}{(2k+1)}4(at+b)^2 + \frac{C_{ij}C_{ij}}{(at+b)^{4k}}\\ &= 2(at+b)^2 + \frac{C_{ij}C_{ij}}{(at+b)^{4k}}\end{aligned}$$

giving $C_{ij}C_{ij} = 0$ and this implies $C_{ij} = 0$

A more general approach to the solution of (28) is now considered, starting with

$$\frac{d^2 J_2'^{k+1}}{dt^2} = AJ_2'^k$$

When k is small, the solution of J'_2 approaches a quadratic equation and for k large, the solution approaches hyperbolic sinh and cosh functions. By letting J'_2 be g^2, we have

$$g\frac{d^2 g}{dt^2} + (2k+1)\left(\frac{dg}{dt}\right)^2 = \frac{A}{2k+2} \tag{42}$$

By letting

$$z = \frac{dg}{dt} \tag{43}$$

we have

$$gz\frac{dz}{dg} = \frac{A}{2k+2} - (2k+1)z^2 \tag{44}$$

Integrating on both sides,

$$\frac{-1}{2(2k+1)}\ln\left|\frac{A}{2k+2} - (2k+1)z^2\right| = \ln|g| + \text{const} \tag{45}$$

Rearranging

$$\frac{A}{2k+2} - (2k+1)z^2 = \frac{B}{g^{2(2k+1)}} \tag{46}$$

we have

$$\frac{dg}{dt} = \sqrt{\frac{\frac{A}{2k+2} - \frac{B}{J_2'^{2k+1}}}{2k+1}} \tag{47}$$

or

$$\frac{dJ_2'}{dt} = \sqrt{\frac{\frac{2AJ_2'}{k+1} - \frac{4B}{J_2'^{2k}}}{2k+1}} \tag{48}$$

Rearranging again, we have the following integral to solved:

$$\sqrt{\frac{2k+1}{k+1}}\frac{dJ_2'^{k+1}}{dt} = \sqrt{2AJ_2'^{2k+1} - 4B(k+1)} \tag{49}$$

If B is taken to be zero, then one would obtain the solution given in (40). This equation has the form of a binomial differentials. According to the Tchebyscheff's theorem, equation (49) does not fulfil the criterion for analytical solution so the integral is not an elementary function (see e.g. Reference 4, p. 406–407) Substituting back into (36), we obtain a solution for the deviatoric stress that depends only on the value of the second deviatoric stress invariant:

$$s_{ij} = \frac{\Delta e_{ij}(2k+1)}{2G\Delta e_{ij}\Delta e_{ij}}\sqrt{\frac{\frac{2AJ_2'}{k+1} - \frac{4B}{J_2'^{2k}}}{2k+1}} + \frac{C_{ij}}{J_2'^k} \tag{50}$$

In order to calculate the value of deviatoric stress at the end of the strain increment, only the value of J_2' at $t = 1$ is needed. Comparing with equation (40) for checking purpose, we have

$$2J_2' = s_{ij}s_{ij} = \frac{\Delta e_{ij}(2k+1)}{2G\Delta e_{ij}\Delta e_{ij}}\frac{\Delta e_{ij}(2k+1)}{2G\Delta e_{ij}\Delta e_{ij}}\frac{\frac{2AJ_2'}{k+1} - \frac{4B}{J_2'^{2k}}}{2k+1} + \frac{C_{ij}}{J_2'^k}\frac{C_{ij}}{J_2'^k} \tag{51}$$

or

$$2J_2' = 2J_2' - \frac{\frac{B(2k+1)}{G^2\Delta e_{ij}\Delta e_{ij}} - C_{ij}C_{ij}}{J_2'^{2k}} \tag{52}$$

giving

$$B = \frac{C_{ij}C_{ij}G^2\Delta e_{ij}\Delta e_{ij}}{2k+1} \tag{53}$$

The value of C_{ij} can be obtained from the initial value of the deviatoric stress. The solution of equation (48) has been attempted using the symbolic algebraic package MAPLE. No closed-form solution is found yet but only a series solution has been obtained and it is given below:

$$\begin{aligned} J_2'(t) = {} & J_2'(0) + t\sqrt{\frac{\%1}{2k+1}} + \frac{1}{2}\frac{\%3}{(2k+1)(k+1)J_2'(0)^{2k+1}}t^2 \\ & + \frac{2}{3}\frac{Bk\sqrt{\frac{\%1}{2k+1}}}{J_2'(0)^{2k+2}}t^3 - \frac{1}{6}\frac{(12k^2B + 20Bk - 4AJ_2'(0)^{2k+1}k + 8B - 3\%2)Bk}{(2k+1)(k+1)J_2'(0)^{4k+3}}t^4 \\ & + \frac{1}{15}\frac{(24k^2B + 34Bk - 4AJ_2'(0)^{2k+1}k + 12B - 3\%2)Bk\sqrt{\frac{\%1}{2k+1}}}{(2k+1)J_2'(0)^{4k+4}}t^5 + O(t^6) \end{aligned} \tag{54}$$

with

$$\%1 = 2\frac{AJ_2'(0)}{k+1} - \frac{4B}{J_2'(0)^{2k}}$$

$$\%2 = AJ_2'(0)^{2k+1}$$

$$\%3 = 4k^2B + 4Bk + \%2 \tag{55}$$

Alternatively, one can assume that the stress rate instead of the strain rate is constant, although this solution suffers from the possible difficulties of negative plastic strain rate when the incremental strain is large. Assuming that

$$s_{ij} = \Delta s_{ij}t + s_{ij}(0) \tag{56}$$

we have, by definition,

$$J_2'(t) = s_{ij}s_{ij}/2 = (\Delta s_{ij}\Delta s_{ij}t^2 + 2s_{ij}(0)\Delta s_{ij}t + s_{ij}(0)s_{ij}(0))/2 \tag{57}$$

with

$$J_2'(0) = s_{ij}(0)s_{ij}(0)/2 \tag{58}$$

Differentiating the second deviatoric stress invariant with respect to time:

$$\mathrm{d}J_2' = (\Delta s_{ij}\Delta s_{ij}t + s_{ij}(0)\Delta s_{ij})\mathrm{d}t \tag{59}$$

Substituting back into equation (22), the rate of plastic work done can be expressed as

$$\mathrm{d}W^s = \frac{H+2G}{2GH}(\Delta s_{ij}\Delta s_{ij}t + s_{ij}(0)\Delta s_{ij})\mathrm{d}t \tag{60}$$

Putting equations (18), (56) and (60) into equation (20), a definite integral relating increment strain with the increment stress rate can be found:

$$\Delta s_{ij}\mathrm{d}t = 2G\mathrm{d}e_{ij} - \frac{2G^2(\Delta s_{ij}t + s_{ij}(0))\dfrac{H+2G}{2GH}(\Delta s_{ij}\Delta s_{ij}t + s_{ij}(0)\Delta s_{ij})\mathrm{d}t}{J_2'(H+2G)} \tag{61}$$

Simplifying,

$$2G\frac{\mathrm{d}e_{ij}}{\mathrm{d}t} = \Delta s_{ij} + \frac{2G(\Delta s_{ij}t + s_{ij}(0))(\Delta s_{ij}\Delta s_{ij}t + s_{ij}(0)\Delta s_{ij})}{H(\Delta s_{ij}\Delta s_{ij}t^2 + 2s_{ij}(0)\Delta s_{ij}t + s_{ij}(0)s_{ij}(0))} \tag{62}$$

For a given deviatoric strain increment Δe_{ij} acting on the Gauss point from $t = 0$ to $t = 1$, we have

$$\Delta e_{ij} = \frac{1}{2G}\int_0^1 (\Delta s_{ij} + \frac{2G(\Delta s_{ij}t + s_{ij}(0))(\Delta s_{ij}\Delta s_{ij}t + s_{ij}(0)\Delta s_{ij})}{H(\Delta s_{ij}\Delta s_{ij}t^2 + 2s_{ij}(0)\Delta s_{ij}t + s_{ij}(0)s_{ij}(0))}\mathrm{d}t \tag{63}$$

For a typical finite element analysis, the only unknown in equation (63) is Δs_{ij} and it can be obtained using a trial and error method. However, as the number of trials would be unknown at the outset, this solution is less suitable for vector computer implementation.

CONCLUSIONS

A series solution of the von Mises elastoplastic model has been presented under a constant strain rate assumption. A trial and error solution is also presented under the assumption of constant stress rate. These solutions should be useful in the assessment of accuracy of numerical integration and also in finite element implementation of the model.

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